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SHORTER COMMUNICATIONS

A SOLUTION TECHNIQUE FOR MULTIDIMENSIONAL ABLATION PROBLEMS

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THE PAPER presents an analytical technique which produces usful closed form solutions for multidimensional ablation (or sublimation) of a solid resulting from surface heating with instantaneous removal of melt.

Consider the half space $z > 0$, initially solid, to be heated on the boundary $z = 0$ by a heat input $H(x, y, t)$. Assume that the temperature of the solid reaches for the first time the melting temperature T_m at $t = 0$. The heat input continues to be applied after $t = 0$, and the melting, whose boundary is given by $S(x, y, t)$, spreads both along the surface and into the body. The melt is assumed here to be instantly removed upon formation. The mathematical formulation [1, 2] of the problem after the start of melting consists of Fourier heat conduction equation,

$$\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t}, \qquad z > S(x, y, t), \tag{1}$$

the moving boundary conditions at $z = S(x, y, t)$,

$$T(x, y, z, t)_{z=S} = T_m, \tag{2}$$

$$\frac{H(x, y, t)}{K} + \left[1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right]\left(\frac{\partial T}{\partial z}\right)_{z=S} = a\frac{\partial S}{\partial t}, \tag{3}$$

where ρ is the density, K the conductivity, C_p the heat capacity per unit mass, l the latent heat of fushion, α the diffusivity $(= K/\rho C_p)$, and $\alpha = \rho l/K$.

The method of solution is first to assume that the heat, conduction equation holds at the phase boundary $(z = S)$. Since at the boundary the temperature is fixed at the melting point, equation (3) yields

$$\frac{\partial T}{\partial \xi} = -\frac{\partial S}{\partial \xi}\frac{\partial T}{\partial z}, \qquad \xi = x \text{ or } y, \tag{4a}$$

$$\frac{\partial T}{\partial t} = -\frac{\partial S}{\partial t}\frac{\partial T}{\partial z}. \tag{4b}$$

Then we find from equations (1) and (4) that the heat conduction equation at the boundary is given by

$$\frac{\partial^2 S}{\partial x^2}\left(\frac{\partial T}{\partial z}\right)_{z=S} - \left[1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right]\left(\frac{\partial^2 T}{\partial z^2}\right)_{z=S} = \frac{1}{\alpha}\frac{\partial S}{\partial t}\left(\frac{\partial T}{\partial z}\right)_{z=S} \tag{5}$$

Combining equations (3) and (5), we see that at $z = S$,

$$\frac{H(x, t)}{K} + \left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} - \frac{1}{\alpha}\frac{\partial S}{\partial t}\right]\left[\left(\frac{\partial T}{\partial z}\right)^2 \Big/ \left(\frac{\partial^2 T}{\partial z^2}\right)\right]_{z=S} = a\frac{\partial S}{\partial t}. \tag{6}$$

We immediately observe that if the temperature term,

$$\left[\left(\frac{\partial T}{\partial z}\right)^2 \Big/ \frac{\partial^2 T}{\partial z^2}\right]_{z=S}$$

is assumed to be either a function of independent variables, or just a constant, a tremendous simplification results. That is, the equation describing the motion of the phase boundary becomes linear and the interface may even be determined a priori for a known heat input.

An obvious candidate for the temperature function T is

$$T = T_m + \sum_{k=1}^{n} \phi_k(x, y, t) f_k(k - S). \tag{7}$$

There are two ways to determine the functional forms of ϕ_k and f_k. But the principle aim is not so much the exactness but the analytical solution of phase interface which is useful for practical purposes. The first method is to introduce the concept of a thermal boundary layer $(S < z < \delta)$ such that beyond δ there is temperature equilibrium and no heat transfer. For example, it may be specified as follows,

$$[T]_{z=\delta} = T_0 \text{ and } \left[\frac{\partial^k T}{\partial z^k}\right]_{z=\delta} = 0, \text{ for } k = 1, \dots m. \tag{8}$$

This technique does not attempt to satisfy the equation of heat conduction and the initial temperature distribution. The second method is to choose the functions f_k and ϕ_k so that T somewhat satisfies equation (1).

A THERMAL LAYER APPROXIMATION

In this approximation it is useful to normalize f_k as follows,

$$T = T_m + \sum_{k=1}^{n} \phi_k f_k\left(\frac{z - S}{\delta - S}\right) \tag{9}$$

The form equations (2), (8) and (9), we find the following algebraic equations:

$$\sum_{k=1}^{n} \phi_k f_k(0) = 0, \tag{10}$$

$$\begin{bmatrix} f_1(1) & f_2(1) & \dots & f_n(1) \\ f_1(1) & f_2(1) & \dots & f_n(1) \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)}(1) & f_2^{(n-1)}(1) & & f_n^{(n-1)}(1) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \phi_n \end{bmatrix} = \begin{bmatrix} -(T_m - T_0) \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad (11)$$

where

$$f_k^{(m)} = \mathrm{d}^m f(\xi)/\mathrm{d}\xi^m.$$

The coefficient matrix of ϕ_k is a Wronskian of nth order. It is obvious that if T_0 is constant, then ϕ_k must be constant. For the nontrivial solution of ϕ_k, f_k may be selected from a complete set of solutions to the homogeneous equation,

$$f^n(\xi) + g_1(\xi)f^{(n)}(\xi) + \dots g_{n-1}(\xi)f(\xi) = 0,$$

where $g_k(\xi)$ are continuous on an open interval (0, 1). Examples of such a set are ξ_k and $\exp(\xi)$ which are often used in one dimensional heat conduction problems.

As an illustration, we consider the set of polynomial functions.

$$f_k = \left(\frac{z - S}{\delta - S}\right)^k.$$

The solution of equations (13) and (14) is

$$\phi_k = (-1)^k \frac{n!}{(n-k)!\,k!}(T_m - T_0), \quad n \neq 1.$$

The temperature distribution is given by

$$T = (T_m - T_0)\left[1 - \frac{z - S}{\delta - S}\right]^n + T_0.$$

It is an easy task to show that equation (6), the displacement diffusion equation, becomes

$$\frac{1}{\alpha_S}\frac{\partial S}{\partial t} - \left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}\right) = Q(x, y, t) \quad (12)$$

where

$$\frac{1}{\alpha_S} = \frac{1}{\alpha}\left[1 + \frac{a\alpha}{(T_m - T_0)}\left(1 - \frac{1}{n}\right)\right].$$

$$Q = \frac{H}{K(T_m - T_0)}\left(1 - \frac{1}{n}\right).$$

Observing that $(\partial^2 S/\partial z^2)_{z=S} = 0$, we find

$$\frac{1}{\alpha_S}\frac{\partial S}{\partial t} - \nabla^2 S = Q(x, y, t).$$

This equation can be easily translated into cylindrical and spherical coordinates, provided that the heat flux vector is normal to the surface and the velocity of the moving boundary is taken to be in the direction of heat flux vector.

APPROXIMATION BY USE OF HEAT CONDUCTION EQUATION

In this approximation it is assumed that the temperature distribution is given by

$$T = T_m \sum_{k=1}^{n} \phi_k(x) g_k(z - S).$$

The thermal conduction equation yields

$$\alpha \sum_{k=1}^{n} \left\{ g_k \frac{\mathrm{d}^2\phi_k}{\mathrm{d}x^2} + \phi_k g_k'' \left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + 1\right] - \phi_k g_k' \left[\left(\frac{\partial^2 S}{\partial x^2}\right) + \left(\frac{\partial^2 S}{\partial y^2}\right) - \frac{1}{\alpha}\frac{\partial S}{\partial t}\right]\right\} = 0$$

where

$$g_k' = \mathrm{d}g_k(\xi)/\mathrm{d}\xi.$$

In order to illustrate the simplest scheme, we assume again that ϕ_k are constant and that g_k are independent of each other. We see then that

$$g_k = A_k \exp[-q(z - S)] + B_k$$

where

$$q = -\sum_{k=1}^{n} \phi_k g'(0) / \sum_{k=1}^{n} \phi_k g''(0)$$

and A_k and B_k are constant. Incorporating ϕ_k into A_k and B_k, we find

$$T = T_m + \sum_{k=1}^{n} A_k\{\exp[-q(z - S)] - 1\} \quad (13)$$

where it is used that the temperature at the boundary is always fixed at the melting point, T_m.

It is obvious that the solution, equation (13), cannot accommodate as general a boundary as shown in the preceding approximation. But it may be useful to study a motion of phase boundary in a half-space where the temperature at $z(x, y) = z_0$ is kept at a fixed value, $T_0 (< T_m)$. For example if z_0 is located at infinity, the temperature distribution is

$$T = T_m + (T_m - T_0)(e^{-q(z-S)} - 1).$$

In this approximation q may be chosen to satisfy the initial temperature distribution at a point other than at $z = 0$ and $z = z_0$. The equation of motion for the phase boundary is determined from equation (6).

$$\frac{1}{\alpha_S}\frac{\partial S}{\partial t} - \frac{\partial^2 S}{\partial x^2} - \frac{\partial^2 S}{\partial y^2} = Q(x, t)$$

where

$$\frac{1}{\alpha_S} = \frac{1}{\alpha}[1 + a\alpha/\phi], \quad \phi = (T_m - T_0), \text{ and } Q = H(x, t)/K\phi.$$

This equation is identical to equation (12), if n is set to be ∞. Therefore regardless of the approximation used, the problem is reduced to solving a linear differential equation of parabolic type.

In order to illustrate an application of the preceding analysis, we shall discuss a continuously vaporizing surface due to laser irradiation and continual removal of the vaporized material from the surface. To simplify the calculations, it is assumed that the problem is in two space variables and that the heat source is symmetric with respect to the origin, $x = 0$. Then according to the preceding analysis, the motion of the depth of hole is given by [3]

$$S(x, t) = \alpha_S \int_0^t \frac{\mathrm{d}t'}{[4\pi\alpha_S(t - t')]^{\frac{1}{2}}} \int_{-\infty}^{\infty} Q(x', t') \exp\left[-\frac{(x - x')^2}{4\alpha_S(t - t')}\right]\mathrm{d}x, \quad (14)$$

where $S(x, 0) = 0$ and l in α_S needs to be interpreted as the latent heat of vaporization per unit mass.

As an example we consider a pulse with shape given by

$$Q(x, t) = Q_0\delta(x)$$

where Q_0 is a constant. The integration of equation (14) produces

$$\frac{S}{S_{max}} = \left\{\exp\left[-\frac{Q_0^2}{4\pi}X^2\right] - \frac{Q_0}{2}|X|\,\mathrm{erfc}\left[\frac{Q_0|X|}{(4\pi)^{\frac{1}{2}}}\right]\right\}, \quad (15)$$

where $X = x/S_{max}$ and $S_{max} = S(0, t) = Q_0(\alpha_S t/\pi)^{\frac{1}{2}}$.

For large values of $Q_0|X|/2\pi^{\frac{1}{2}}$, the solution (15) becomes [4]

$$\frac{S}{S_{max}} \simeq \frac{2\pi}{Q_0^2X^2}\exp\left(-\frac{Q_0X^2}{4\pi}\right) \simeq \frac{2\pi}{Q_0^2X^2}.$$

Therefore if the surface aperture is measured by the distance at which $S/S_{max} = \varepsilon^2 =$ a small constant, then the radius of the aperture is given by

$$x_\varepsilon \simeq \frac{\sqrt{(2)}}{\varepsilon}(\alpha_S t)^{\frac{1}{2}}.$$

The hole spreads along the surface proportional to $t^{\frac{1}{2}}$. The same conclusion is reached by Boley and Yagoda [1] from their early time solutions. It is also seen that the spreading is a function of α_S only.

$$\alpha_s = \frac{K(T_v - T_0)}{\rho[C_p(T_v - T_0) + l_v(1 - 1/n)]}$$

where T_v is the temperature of boiling point and l_v the latent heat of vaporization per unit mass.

However, contrary to Boley and Vagoda [1], the maximum penetration into the solid is also proportional to $t^{\frac{1}{2}}$. For example, in aluminum, a crater depth of one millimeter may be produced by a millisecond-duration pulse with the heat flux of 15 kW/cm. This value falls in the range of representative values given in Ready [5].

The shape of holes must be computed with a much more realistic heat input than the delta function, but this may require a numerical integration of equation (14). If equation (15) is used, then the slope of the hole at $X = 0 = -Q_0/2$ may be used as a measure for the steepness of the hole. For aluminum and the heat flux of 15 kW/cm, the slope is about -3. This is again comparable with a configuration of hole shown in Ready [5].

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INFLUENCE OF THERMAL PROPERTIES ON FILM COOLING EFFECTIVENESS

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INTRODUCTION

The recent publication of *Analysis of Heat and Mass Transfer* [1] and the re-introduction of a semi-analytical analysis of the film cooling effectiveness of [2] is the motivation for these authors to present certain aspects of the problem not yet available in the literature. Although the effectiveness presented in [1, 2] yields satisfactory agreement with experimental data, it is questionable as to whether the influence of variable fluid properties is expressed correctly [1]. In the present investigation, an attempt is made to study the influencing effect of the fluid properties in a purely analytical manner. The only empirical relation considered is the well-known Prandtl equation relating shear stress to momentum thickness. Also, it has been assumed that a power law relation $u/u_e = (y/\delta)^{1/n}$ for the velocity distribution in the boundary layer holds far downstream from the slot.